MELTING OF A SEMIINFINITELY LARGE BODY BY AN INTERNAL POINT SOURCE OF HEAT

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The propagation of the melt front in a semiinfinitely large solid body with an internal point source of heat is analyzed.

An electron beam acting on a material generates in it a heat source whose intensity is maximum at a certain depth [1, 2] ($\sim 10^{-3}$ cm for electron energies in the 100-150 keV range). Such a mode of energy release results sometimes in local melting of the solid from inside.

In order to analyze the process of liquefaction and the accompanying changes within a solid body, we use the model where the total power of the electron beam is released at a point corresponding to maximum energy generation. Immediately after the source has been turned on, the substances within an infinitesimally small volume around the origin of coordinates will melt (Fig. 1). The manner in which the interphase boundary propagates is determined from the solution to the Stefan problem

$$\frac{1}{\varkappa_1}\frac{\partial T_1}{\partial t} - \nabla^2 T_1 = -\frac{q}{\lambda_1}\delta(r), \tag{1}$$

$$\frac{1}{\varkappa_2}\frac{\partial T_2}{\partial t} - \nabla^2 T_2 = 0, \tag{2}$$

$$T_2(\mathbf{r}, 0) = T_0, \tag{3}$$

$$T_1(\mathbf{a}, t) = T_2(\mathbf{a}, t) = T_m, \tag{4}$$

$$\nabla T_2(\infty, t) = 0, \tag{5}$$

$$\left.\frac{\partial T_2}{\partial z}\right|_{z=0} = 0. \tag{6}$$

At the interphase boundary [3]

$$\rho L\frac{\partial F}{\partial t} + (\lambda_2\nabla T_2 - \lambda_1\nabla T_1, \nabla F)_s = 0, \tag{7}$$

where F(a, t) = 0 is the equation of the interface.

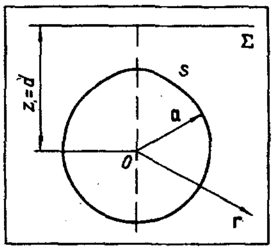

Fig. 1. Schematic diagram showing the propagation of the melt front in a semi-infinitely large body.

We solve problem (1)-(6) approximately, considering ($|a|/d < 1$) the free surface as the perturbation source. (The characteristics of the perturbations will be discussed later.)

<u>Symmetrical Problem.</u> In the zeroth approximation (condition (6) omitted) the problem has spherical symmetry and at a time $t \gtrsim \bar{t}$, when the melting rate is much lower than the heating rate, the approximate solution may be appropriately expressed in terms of the functions [4]

$$T_1 = \frac{q}{4\pi\lambda_1}\left(\frac{1}{r} - \frac{1}{a}\right) + \sum_{n=1}^{\infty}\left\{2a(T_m - T_0)(-1)^n - \frac{q}{4\pi\lambda_1}\right\} \tag{8}$$

$$\times\frac{1}{n}\exp\left[-\frac{\varkappa_1 n^2\pi^2 t}{a^2}\right]\sin\frac{n\pi r}{a} + T_m, \tag{9}$$

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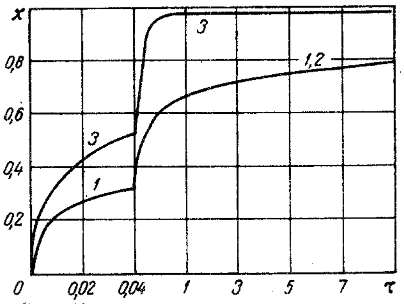

Fig. 2. Location of the interphase boundary, as a function of time.

$$T_2 = \frac{a(T_m - T_0)}{r}\ \mathrm{erfc}\left(\frac{r-a}{2\sqrt{\varkappa_2 t}}\right) + T_0.$$

Disregarding the transient term in the expression for T_1, which estimates have shown to be permissible at $t \approx \bar{t}$, and inserting the expressions for T_1, T_2 into the boundary condition at the interface, we obtain

$$\frac{q}{4\pi a^2} - \lambda\frac{T_m - T_0}{a} - \lambda\frac{T_m - T_0}{\sqrt{\pi\varkappa t}} = \rho L\frac{da}{dt}$$

or

$$q - 4\pi\lambda(T_m - T_0)a - 4\pi\lambda(T_m - T_0)\frac{a^2}{\sqrt{\pi\varkappa t}} = 4\pi\rho L a^2\frac{da}{dt}. \tag{10}$$

In dimensionless form this becomes

$$1 - x - \frac{x^2}{\sqrt{\tau}} = \gamma x^2\frac{dx}{d\tau}. \tag{11}$$

In order to examine the behavior of x(t) near zero ($\tau \ll 1$, $x \ll 1$), we substitute

$$\psi = x^3, \quad \xi = \sqrt{\tau}. \tag{12}$$

Then (11) becomes

$$\frac{d\psi}{d\xi} = \frac{6\xi}{\gamma}(1 - \psi^{1/3}) - \frac{6}{\gamma}\psi^{2/3}. \tag{13}$$

A solution to Eq. (13) exists uniquely at all points $\xi > 0$. At $\xi = 0$ the Lipschitz condition is not satisfied and, as a result, the solution $\psi = 0$ is added here. The Euler method readily yields a solution to (13) with the initial condition $\psi|_{\xi=0} = 0$ in the form

$$\psi = \frac{3}{\gamma}\xi^2 - \frac{6}{\gamma}\left(\frac{3}{\gamma}\right)^{2/3}\frac{3}{7}\xi^{7/3} - \frac{6}{\gamma}\,\frac{3}{8}\left(\frac{3}{\gamma}\right)^{1/3}\xi^{8/3} + \ldots \tag{14}$$

Retaining only the first term in (14) at $t \ll 1$ yields, after a change to variables x and τ,

$$x \approx \sqrt[3]{\frac{3}{\gamma}\tau}. \tag{15}$$

It is easy to show that formula (15) is identical to the relation derived from the equation of energy balance. Consequently, the solution to (11) behaves initially in conformity with the characteristics of the melting process, even though da/dt is not a small quantity.

At $\tau \gtrsim 1$ we have $dx/d\tau \ll 1$, i.e., the right-hand side of (11) $\gamma x^2 dx/d\tau \ll 1$. Solving Eq. (11) by the iteration method yields

$$x_0 = \frac{\sqrt{\tau}}{2}\left(\sqrt{1 + \frac{4}{\sqrt{\tau}}} - 1\right), \tag{16}$$

$$x_1 = \frac{\sqrt{1 + 4\left(\frac{1}{\sqrt{\tau}} + \frac{\gamma dx_0}{d\tau}\right)} - 1}{2\left(\frac{1}{\sqrt{\tau}} + \gamma\frac{dx_0}{d\tau}\right)}, \tag{17}$$

. . . .

$$x_n = \frac{\sqrt{1 + 4\left(\frac{1}{\sqrt{\tau}} + \frac{\gamma dx_{n-1}}{d\tau}\right)} - 1}{2\left(\frac{1}{\sqrt{\tau}} + \gamma\frac{dx_{n-1}}{d\tau}\right)}. \tag{18}$$

At $\tau \gg 1$ the term $4(1/\sqrt{\tau} + \gamma dx_{n-1}/d\tau)$ becomes much smaller than unity and, accurately down to second-order terms,

$$x \approx 1 - \frac{1}{\sqrt{\tau}}. \tag{19}$$

A numerical solution to Eq. (11) with $\gamma = 1$ is shown in Fig. 2 (curve 1) (according to calculations, at $\tau > 0$ the parameter γ has almost no effect on the trend of the curve). Curve 2 represents formula (16),

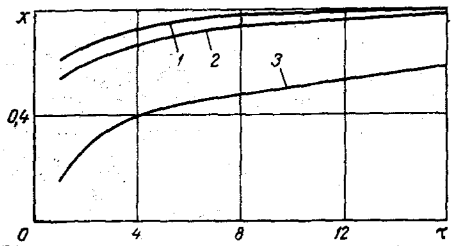

Fig. 3. Location of the interphase boundary, with a distributed heat source, as a function of time: r/b = 8 (1), 4 (2), 2 (3).

and curve 3 represents the quasisteady case. A comparison indicates that the propagation of the melt front at $\tau \geq 1$ is governed by the transiency of heat conduction.*

Semiinfinitely Large Body. Inasmuch as energy losses on phase transformation are negligible or not significant at $\tau \geq 1$, it may be assumed that introducing a free surface will affect the melt front in the same way as the temperature field, i.e., that

$$T_1 \approx \frac{q}{4\pi\lambda}\left(\frac{1}{r} + \frac{1}{\sqrt{r^2 - 4rd\cos\theta + 4d^2}} - \frac{1}{\eta}\right) + T_m, \tag{20}$$

$$T_2 \approx T_0 + (T_m - T_0)\eta\left(\frac{1}{r} + \frac{1}{\sqrt{r^2 - 4rd\cos\theta + 4d^2}}\right)\operatorname{erfc}\left(\frac{r - r(\eta)}{2\sqrt{\varkappa t}}\right). \tag{21}$$

At the melt front, moreover,

$$\frac{1}{\eta} = \frac{1}{r} + \frac{1}{\sqrt{r^2 - 4rd\cos\theta + 4d^2}} \tag{22}$$

or, what is equivalent,

$$r = \eta\left(1 + \frac{r}{\sqrt{r^2 - 4rd\cos\theta + 4d^2}}\right). \tag{23}$$

At $r/2d < 1$ expression (23) resolves into Legendre polynomials and becomes

$$r = \eta\left[1 + \sum_{n=0}^{\infty}\left(\frac{r}{2d}\right)^{n+1} P_n(\cos\theta)\right] = f(r). \tag{24}$$

Since $f'(r) < 1$, hence r as a function of η is calculated by iteration:

$$r_0 = \eta, \tag{25}$$

$$r_1 = \eta\left(1 + \frac{\eta}{2d}\right), \tag{26}$$

$$r_2 = \eta\left[1 + \frac{\eta}{2d} + \left(\frac{\eta}{2d}\right)^2 (1 + \cos\theta)\right]. \tag{27}$$

Inserting (20), (21), and (22) into the condition of heat balance at a moving boundary, and integrating with respect to $\eta/2d = \beta$ over the interface accurately down to second-order terms, we obtain an equation in $y = \eta/r$:

$$1 - y - \frac{y^2}{\sqrt{\tau}}(1 + 2\beta + 3\beta^2) = \gamma y^2 \frac{dy}{d\tau}(1 + 4\beta + 10\beta^2). \tag{28}$$

Here, as also in (11), $y_0 = x_0$ with $\gamma\, dy/d\tau$ being much smaller than $1/\sqrt{\tau}$. The result of subsequent iteration with a free surface introduced here is, unlike (17),

$$y_1 = \frac{\sqrt{1 + 4\left[\frac{1}{\sqrt{\tau}}(1 + 2\beta) + \gamma\frac{dy_0}{d\tau}\right]} - 1}{2\left[\frac{1}{\sqrt{\tau}}(1 + 2\beta) + \gamma\frac{dy_0}{d\tau}\right]}. \tag{29}$$

Simple transformations will reduce expression (29) to the form

$$y_1 = y_0(1 - A) + B, \tag{30}$$

where

$$A = \left(2\beta + \gamma\sqrt{\tau}\frac{dy_0}{d\tau}\right)\frac{1 - 2/\sqrt{\tau}}{1 + 4/\sqrt{\tau}}, \tag{31}$$

$$B = \frac{2\beta + \gamma\sqrt{\tau}\frac{dy_0}{d\tau}}{1 + 4/\sqrt{\tau}}. \tag{32}$$

*As this article was nearing completion, the authors learned about the study in [5] concerning the kinetics of liquefaction due to a point source on the surface. The authors of [9], using the solution given there in terms of a series in $(r - a(t))$, disregarded all terms except the zeroth one in the boundary condition at infinity and, therefore, obtained the solution for $a(t)$ in the quasisteady approximation.

Equation (30) indicates clearly that y_1 increases with time somewhat slower than x, while the correction to x decreases. Taking into account that $\eta_0 = a$, we obtain for the radius of the melt zone

$$r_1 = a(1 + \beta - A) + B. \tag{33}$$

Relation (33) and subsequent iterations indicate the propagation characteristics of the melt front in the presence of a free surface at $\tau > 1$ and with $\beta \ll 1$.

Effect of Source Distribution. A point source represents a convenient idealization of a practically realizable lumped source whose intensity may have, for example, the following space distribution:

$$u(r) = \frac{B}{\lambda r} \exp\left\{-\frac{r}{b}\right\}. \tag{34}$$

Normalization yields $B = q/4\pi b^2$.

Unlike with a point source, melting due to a volume source begins not immediately after turn-on but some time after the maximum temperature within the given region has reached the melting point. For the problem with spherical symmetry and with the source (34) we have

$$T_{\max} = T_{r=0} = \frac{q}{4\pi\lambda b}\left[1 - \exp\left(\frac{\varkappa t}{b^2}\right) \operatorname{erfc}\left(\sqrt{\frac{\varkappa t}{b^2}}\right)\right] + T_0. \tag{35}$$

Equating T_{max} to the melting point, we readily find from (35) the instant of time at which melting will begin. The temperature field at the instant when $T_{max}(t) = T_m$ represents the initial condition for the melting problem. Disregarding the space variation of the initial temperature (which is reasonable because, in the final analysis, many process characteristics depend on T_0 [6]), we obtain condition (7) with the point source (34) as

$$\frac{q}{4\pi a^2}\left[1 - \exp\left(-\frac{a}{b}\right)\right] - \frac{q}{4\pi ab} \exp\left(-\frac{a}{b}\right) \exp\left(\frac{\varkappa t}{b^2}\right) \operatorname{erfc}\left(\frac{\sqrt{\varkappa t}}{b}\right) - \frac{\lambda(T_m - T_0)}{\sqrt{\pi\varkappa t}} - \frac{\lambda(T_m - T_0)}{a} = \rho L \frac{da}{dt} \tag{36}$$

or in dimensionless form

$$1 - \exp\left(-\frac{\bar{r}}{b}x\right) - \frac{\bar{r}}{b}x \exp\left(-\frac{\bar{r}}{b}x\right) \exp\left(\frac{\bar{r}^2}{b^2}\tau\right) \operatorname{erfc}\left(\frac{\bar{r}}{b}\sqrt{\tau}\right) - x - \frac{x^2}{\sqrt{\tau}} = \gamma x^2 \frac{dx}{d\tau}. \tag{37}$$

When $r/b\sqrt{\tau} > 1$,

$$\exp\left(\frac{\bar{r}^2}{b^2}\tau\right) \operatorname{erfc}\left(\frac{\bar{r}}{b}\sqrt{\tau}\right) \approx \frac{b}{\bar{r}\sqrt{\tau}}$$

and relation (37) becomes

$$1 - \exp\left(-\frac{\bar{r}}{b}x\right) - \frac{x}{\sqrt{\tau}} \exp\left(-\frac{\bar{r}}{b}x\right) - x - \frac{x^2}{\sqrt{\tau}} = \gamma x^2 \frac{dx}{d\tau}. \tag{38}$$

The dimensionless parameter $\bar{r}/b$ determines the effect of source distribution on the characteristics of the melting process. When the ratio of maximum melt radius with an internal source to the dimension of the zone of intensive heat generation remains $r/b \gg 1$, then it may be regarded as a point curve. For $r/b \lesssim 1$ the source distribution is taken into account.

Numerical solutions to Eq. (38) are shown in Fig. 3 for $\bar{r}/b = 8$, 4, and 2.

Stress Field. In order to estimate the time till spillage occurs, which depends on the pressure inside the melt and on the dimensions of the plasticity zone, it is necessary to determine the stress field produced in the solid body by the nonuniform temperature field [8] as well as by the difference between specific volumes in the liquid state and in the solid state respectively [7]. We will analyze the problem under the following assumptions:

1. The melt boundary propagates at a velocity much lower than the velocity of sound in the given substance;

2. the temperature field is quasisteady;

$$T = T_m a(t)/r;$$

3. the plastic properties of the substance are described by an ideally plastic body with the yield point depending on the temperature.

As a consequence of spherical symmetry, shear strains $\varepsilon_{r\varphi}$, $\varepsilon_{\varphi\theta}$, $\varepsilon_{r\theta}$ and tangential stresses are equal to zero, while strains $\varepsilon_{\varphi\varphi} = \varepsilon_{\theta\theta} = \varepsilon_\varphi$ and $\sigma_{\varphi\varphi} = \sigma_{\theta\theta} = \sigma_\varphi$.

The "radial" component of the stress tensor σ_r and the "angular" component of the strain tensor both satisfy the boundary condition

$$\begin{aligned} \sigma_r &= 0 \quad \text{at} \quad r \to \infty, \\ \sigma_r &= -p \quad \text{at} \quad r = a, \\ \varepsilon_\varphi &= \varepsilon_0 - k_l p \quad \text{at} \quad r = a. \end{aligned} \tag{39}$$

Using the equations of equilibrium and continuity as well as the known relations between stress and strain tensors for the elastic zone and the plastic zone respectively, we obtain the following expressions for the pressure in the liquid and for the radius of the plastic zone

$$p = \frac{2}{3}\sigma_s^R\left[1 + \frac{3}{\sigma_s^R}\int_a^R \frac{\sigma_s}{r}\,dr + 3\,\frac{\alpha G T_m a}{\sigma_s^R R}\cdot\frac{1+\nu}{1-\nu}\right], \tag{40}$$

$$\frac{R^3}{a^3}\left(\frac{1}{4G} + k_s\right)\left(\frac{2}{3}\sigma_s^R + \frac{4}{3}\alpha G T_m\frac{1+\nu}{1-\nu}\frac{a}{R}\right) + (k_l - k_s)\,p = \varepsilon_0 - \frac{3}{2}\alpha T_m\left(\frac{R^2}{a^2} - 1\right), \tag{41}$$

where $\sigma_s^R = \sigma_s(R)$.

As in the case considered in [7], the pressure in the liquid is independent of the melt radius (the result of a self-adjoint problem).

If the yield point σ_s near the melting point is a following function of the temperature

$$\sigma_s = \sigma_s^0\,\frac{T_m - T}{T}$$

(the constant σ_s^0 being determined from the test curve in [10]), then estimates indicate that the pressure is approximately equal to the yield point under normal conditions in a melt and that the radius of the plasticity zone is almost equal to the radius of the melt zone. For example, the pressure in an aluminum melt is $p \approx 0.80 \cdot 10^4$ atm and $R/a \approx 1.1$

It is to be noted that all these results are valid for heat sources whose intensity is limited by the condition $\bar{r}/\bar{d} \lesssim 1$. When $\bar{r}/\bar{d} \gg 1$, i.e., when the maximum melt radius is much greater than the distance from the point of peak energy release to the surface, then the assumption concerning the perturbation of melt isotherms is incorrect because, as the melt front approaches the surface, $\gamma\, dx/d\tau \gtrsim 1/\sqrt{\tau}$.

NOTATION

T	is the temperature, °K;
c	is the specific heat, cal/g · °C;
$\varkappa$	is the thermal diffusivity, cm²/sec;
λ	is the thermal conductivity, cal/cm · sec · °C;
q	is the heat generation rate, cal/sec;
r	is the radius vector of a local point within the analyzed region;
s	is the surface of interphase boundary;
Σ	is the free plane surface;
z	is the distance from the free surface, along a normal;
∇^2	is the three-dimensional Laplace operator;
∇	is the gradient operator;
d	is the distance from free surface to point of maximum energy release;
$\bar{r} = q/4\pi\lambda(T_m - T_0)$	is the maximum radius of melt zone, with a given power q of a constant point source;
$\bar{t} = \bar{r}^2/\pi\varkappa$, $t_1 = 4\pi\rho L\bar{r}^3/q$	are the characteristic times associated with the transiency in the external problem of heat conduction and a moving interphase boundary;
$x = r/\bar{r}$, $\tau = t/\bar{t}$	is the dimensionless radius and time;
$1/\gamma = t_1/\bar{t} = (T_m - T_0)c/\pi L$	is the dimensionless parameter;
η	is the parameter of the melt isotherm, cm;
b	is the parameter characterizing the source distribution;
r, φ, θ	are the spherical coordinates;
p	is the pressure in the melt;
$3k_l$, $3k_s$	are the volume compressibility of liquid and of solid respectively;
$\sigma_s = \sigma_s(T)$	is the yield point of the substance;

R is the radius of plasticity zone;
G is the shear modulus;
ν is the Poisson ratio;
α is the linear expansivity of the solid;
$3\varepsilon_0$ is the change in the specific volume of the substance during melting.

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